

Euclidean Majorana and Dirac Fields

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Abstract: Euclidean Majorana fields are constructed within the Berezin calculus and as random fields on the measure space of fermionic white noise. These fields are covariant under Euclidean transformations, and their expectation values reproduce the Schwinger functions of Majorana fermions in $d = 2, 3, 4 \bmod 8$ dimensions. Euclidean Dirac fields and their conjugate fields can be obtained as linear functions of two independent Majorana fields by equations known from classical spinor fields on the Minkowski space.

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I. Introduction:

The Euclidean quantum field theory of fermions can be based on the algebra of skew symmetric forms which generate the Schwinger functions (Euclidean Green's functions). But it is instructive to go beyond that frame and to introduce Euclidean spinor fields. The n -point functions of these fields, or a subset of the n -point functions, have to coincide with the Schwinger functions. The first explicit example of Euclidean Dirac operators defined on a Fock space has been given by Osterwalder and Schrader [1]. A modified version which also includes Majorana fermions can be found in [2]. Later Nicolai [3] and then Borthwick [4] presented Euclidean Majorana fields which are closer to the original construction of Osterwalder and Schrader. In [1] Osterwalder and Schrader introduced two independent anticommuting Euclidean fields corresponding to the relativistic fields ψ and $\bar{\psi}$. These extra degrees of freedom also entered the Berezin functional integration [5], but in the calculations of fermionic Green's functions one usually does not make an explicit statement about the independent degrees of freedom. Since

arguments based on Euclidean fields, mainly in the Berezin formulation, have become a standard technique, the dependence of fields should be taken correctly into account. The importance of this point can clearly be seen in a recent publication of Haridas Banerjee and his collaborators [6].

In this article the problem of Euclidean fields for fermions is investigated on the basis of random fields with an anticommuting multiplication [7,8]. The basic algebraic structure is a Grassmann algebra, and not a Clifford algebra as in the Fock space constructions mentioned above. It is therefore possible to perform the algebraic part of the calculations for random fields and for Berezin fields in a parallel way. To investigate the relations between Majorana fields, the Dirac field and the conjugate (adjoint) Dirac field it is sufficient to consider free fermions. In the following sections we present a theory of free Euclidean Majorana and Dirac fields with positive mass in $d = 4$ dimensions. The results can be immediately transferred to $d = 2, 3, 4 \bmod 8$ dimensions, and the case of mass zero fields needs only a slight technical modification concerning the test function spaces. After a short recapitulation of the Schwinger functions of Majorana fermions in Sect. 2 the basic fields, the white noise fields and their Berezin counterparts, are presented in Sect. 3. For the construction of Majorana fields we need fermionic real white noise fields which are covariant under the Euclidean group. In Sect. 4 Majorana and Dirac fields are defined as linear transforms of white noise fields. A remarkable result is that the Euclidean Dirac field ψ_D

and its conjugate $\tilde{\psi}_D$ can be obtained from two independent Majorana fields $\psi_{M(j)}$, $j = 1, 2$, by equations known from classical spinor fields on the Minkowski space, i.e.

$$\begin{cases} \psi_D = \psi_{M(1)} + i \psi_{M(2)}, \\ \tilde{\psi}_D = C \psi_{M(1)} - i C \psi_{M(2)}. \end{cases} \quad (1)$$

Here C is the real unitary charge conjugation matrix which satisfies

$$C^T = -C \text{ and } C \gamma_\mu C^T = -\gamma_\mu^T \quad (2)$$

for all Hermitean Dirac matrices γ_μ , $\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2\delta_{\mu\nu}$, $\mu, \nu = 1, 2, \dots, d$. Such a charge conjugation matrix can be constructed for $d = 2, 3, 4 \bmod 8$ dimensions, and all arguments apply to these

dimensions. For $d = 2, 3 \bmod 8$ there exists a real representation of the Euclidean Dirac matrices. Then the Majorana fields are real fields, and (1) gives the decomposition of the Dirac field into its real and its imaginary part. For $d = 4 \bmod 8$ the Majorana fields are still real with respect to a non-local involution, and (1) is the decomposition into the real and the imaginary part with respect to this involution. The representation (1) clearly shows that the conjugate field $\tilde{\psi}_D$ can be calculated from the Dirac field ψ_D .

II. Schwinger functions of the Majorana field:

Let $\Psi_\alpha(\tilde{x})$, $\alpha = 1, \dots, 4$, be a free Majorana field operator on the Minkowski space M^4 , $\tilde{x} \in M^4$, then the 4×4 matrix of its τ -function (Feynman propagator)

$$\mathcal{F}_{\alpha\beta}(\tilde{x}, \tilde{y}) = \langle \text{vac} | T \Psi_\alpha(\tilde{x}) \Psi_\beta(\tilde{y}) | \text{vac} \rangle$$

$$\alpha, \beta = 1, \dots, 4, \quad \hat{x}, \hat{y} \in M^4 \quad (3)$$

can be analytically continued into the Euclidean region $x^k = \tilde{x}^k$, $y^k = \tilde{y}^k$, $k = 1, \dots, 3$ and $x^4 = -i\tilde{x}^0$, $y^4 = -i\tilde{y}^0$, to the matrix of the two-point Schwinger functions

$$M_{\alpha\beta}(x, y) = \sum_{\gamma} S_{\alpha\gamma}(x, y) C_{\gamma\beta} \quad (4)$$

Here the matrix $S(x, y)$ is the usual two-point Schwinger function of Dirac fermions

$$S(x, y) = (2\pi)^{-4} \int \frac{m - i\not{p}}{m^2 + p^2} e^{-ip(x-y)} d^4 p \quad (5)$$

and C is the real unitary charge conjugation matrix (2).

For the Euclidean variables we adopt the notations

$$x = (x^1, x^2, x^3, x^4) \in \mathbb{R}^4, \quad p x = \sum_{\mu=1}^4 p^\mu x^\mu, \quad \not{x} = \sum_{\mu} \gamma_\mu \frac{\partial}{\partial x^\mu}.$$

The matrix function (4) is the inverse of the differential operator $C^T(m-\partial)$. Since the Euclidean Dirac operator $(m-\partial)$ satisfies $C^T(m-\partial) = -(m-\partial)^T C$, we immediately realize that (4) is a non-degenerate skew symmetric matrix function

$$M_{\alpha\beta}(x,y) = -M_{\beta\alpha}(y,x) \quad (6)$$

which can be taken as kernel of a bilinear skew symmetric form on the Hilbert space $\mathcal{X} = \mathcal{L}^2(\mathbb{R}^4) \otimes \mathbb{C}^4$. The symplectic form (= non-degenerate bilinear skew symmetric form) of the two-point functions is then

$$\begin{aligned} \omega_M(f,g) &= \sum_{\alpha,\beta=1}^4 \int f(\alpha,x) M_{\alpha\beta}(x,y) g(\beta,y) d^4x d^4y \\ &= \int f^T(x) S(x,y) C g(y) d^4x d^4y \end{aligned} \quad (7)$$

with $f,g \in \mathcal{X}$. For positive mass $m > 0$ this functional is continuous in the norm of \mathcal{X} . For mass zero one should take smooth rapidly decreasing test functions from the Schwartz space $\mathcal{S} = \mathcal{S}(\mathbb{R}^4) \otimes \mathbb{C}^4$.

The n -point Schwinger function of the Majorana fermion $M^{(n)}(\alpha_1, x_1, \alpha_2, x_2, \dots, \alpha_n, x_n)$, $\alpha_j = 1, \dots, 4$, $x_j \in \mathbb{R}^4$, $j = 1, \dots, n$, is calculated in the same way as the n -point τ -function with the fermionic Gaussian combinatorics, see e.g. [9]

$$\begin{aligned} M^{(n)} &= 0 \quad \text{if } n \text{ odd,} \\ M^{(2n)}(\alpha_1, x_1, \dots, \alpha_{2n}, x_{2n}) &= \text{pf}(M_{\alpha_i \alpha_j}(x_i, x_j)), \\ n &= 1, 2, \dots \end{aligned} \quad (8)$$

The pfaffian $\text{pf } A$ of a $2n \times 2n$ matrix A_{ij} is defined as

$$\text{pf } A = \frac{1}{2^n n!} \sum_{\sigma} \text{sign } \sigma A_{\sigma(1)\sigma(2)} \dots A_{\sigma(2n-1)\sigma(2n)}. \quad (9)$$

The summation extends over all permutations σ of the numbers $\{1, \dots, 2n\}$. For the following arguments it is convenient to smear the

locally singular Schwinger functions with test functions from \mathcal{X} (or from \mathcal{S}). The corresponding Schwinger functional is then evaluated with (7) as

$$\begin{aligned} \sum_{\alpha} \int M^{(2n)}(\alpha_1, x_1, \dots, \alpha_{2n}, x_{2n}) f_1(\alpha_1, x_1) \dots f_{2n}(\alpha_{2n}, x_{2n}) dx \\ = \text{pf}(\omega_M(f_i, f_j)). \end{aligned} \quad (10)$$

Schwinger functions for Majorana fermions can be defined for all dimensions d for which a charge conjugation matrix (2) exists, i.e. for $d = 2, 3, 4 \pmod{8}$. In all these cases a theory of Majorana fields on the Minkowski space can be reconstructed from the Schwinger functions following the arguments of Osterwalder and Schrader [10]. More details about Osterwalder–Schrader positivity and reconstruction for Majorana fermions can be found in [11]. For the reconstruction of the physical theory it is necessary that the test function space is a complex linear space. The choice of the complex test function spaces \mathcal{X} (or \mathcal{S}) in the equations above is therefore not only of technical convenience – the Schwinger function (5) has a real representation only for $d = 2, 3 \pmod{8}$ – but it is essential for the reconstruction.

III. Fermionic white noise:

Real four component white noise $\xi^\alpha(x)$, $\alpha = 1, \dots, 4$, on \mathbb{R}^4 is a random field with Gaussian measure $d\mu$ on the space of real tempered distributions $\mathcal{S}'_{\mathbb{R}}(\mathbb{R}^4) \otimes \mathbb{R}^4$. Since $\xi^\alpha(x)$ is locally singular it is convenient to smear $\xi^\alpha(x)$ with test functions from the real Hilbert space $\mathcal{X}_{\mathbb{R}} = \mathcal{L}^2_{\mathbb{R}}(\mathbb{R}^4) \otimes \mathbb{R}^4$ (or even with infinitely differentiable strongly decreasing functions from $\mathcal{S}_{\mathbb{R}}(\mathbb{R}^4) \otimes \mathbb{R}^4$). The variables

$$\xi(f) = \sum_{\alpha=1}^4 \int f(\alpha, x) \xi^\alpha(x) d^4x \quad (11)$$

with $f \in \mathcal{X}_{\mathbb{R}}$ are then integrable Gaussian variables on the measure space $\mathcal{S}'_{\mathbb{R}}(\mathbb{R}^4) \otimes \mathbb{R}^4$. The expectation values (mean values) of func-

tionals $\phi(\xi)$ of white noise, e.g. a monomial $\phi(\xi) = \xi(f_1) \dots \xi(f_n)$, $f_i \in \mathcal{X}_{\mathbb{R}}$, are denoted by $E\phi(\xi) \equiv \int \phi(\xi) d\mu(\xi)$. Since the measure is Gaussian all expectation values follow from

$$E \xi(f) = 0 \text{ and}$$

$$E \xi(f) \xi(g) = \langle f | g \rangle \quad (12)$$

Here $\langle f | g \rangle = \sum_{\alpha} \int f(\alpha, x) g(\alpha, x) d^4 x$ is the positive definite inner product of $\mathcal{X}_{\mathbb{R}}$. In [12] it has been shown that it is possible to define an anticommutative multiplication for the variables $\xi(f)$, in the following denoted by $\xi(f) \star \xi(g)$, with the properties:

- The product is isomorphic to the antisymmetric tensor product.
- The expectation values of monomials (with this multiplication) are calculated as

$$E \xi(f_1) \star \dots \star \xi(f_n) = 0 \quad \text{if } n \text{ odd}$$

$$E \xi(f_1) \star \dots \star \xi(f_{2n}) = \text{pf } (\omega_0(f_i, f_j)) \quad (13)$$

with the symplectic form

$$\omega_0(f, g) = \sum_{\alpha, \beta} \int f(\alpha, x) C_{\alpha\beta} g(\beta, x) d^4 x \quad (14)$$

where C is a real orthogonal matrix which satisfies $C^2 = -\text{id}$.

White noise provided with this anticommutative multiplication will be denoted as fermionic white noise. This product is closely related to the non-commutative calculus of Hudson and Parthasarathy [13]. Eq. (14) shows that the expectation values of polynomials of fermionic white noise follow with the usual fermionic Gaussian combinatorics from the two-point functional $E \xi(f) \star \xi(g) = \omega_0(f, g)$, or, equivalently, from the two-point function

$$E \xi^{\alpha}(x) \star \xi^{\beta}(y) = C_{\alpha\beta} \delta^4(x-y) \quad (15)$$

For the subsequent construction of Euclidean Majorana fields we need a fermionic white noise which is covariant under Euclidean

transformations. For that purpose it is necessary that the skew symmetric form (14) is invariant against Euclidean transformations, the test functions taken as Euclidean spinor fields. This property is guaranteed if we choose the charge conjugation matrix (2) as the antisymmetric matrix in (14). The generators of spinorial rotations are $\gamma_\mu \gamma_\nu$, $\mu \neq \nu$, and the charge conjugation matrix (2) satisfies

$C\gamma_\mu \gamma_\nu + (\gamma_\mu \gamma_\nu)^T C = 0$ if $\mu \neq \nu$. For more details about Euclidean transformations see e.g. [1] and, concerning random fields, [8].

With respect to the algebraic properties there is a great similarity between fermionic random fields and fermionic Berezin fields. Let $\mathcal{H}_\mathbb{R} = \mathcal{L}^2_\mathbb{R}(\mathbb{R}^4) \otimes \mathbb{R}^4$ be the real Hilbert space which we have used before. The "Berezin field" $\hat{\xi}^\alpha(x)$ is an isometric mapping of $\mathcal{H}_\mathbb{R}$ onto a real Hilbert space \mathcal{H}_1 which generates a Grassmann algebra \mathcal{G} , see [14] Sect. 3,

$$f \in \mathcal{H}_\mathbb{R} \rightarrow \hat{\xi}(f) = \sum_\alpha \int \hat{\xi}^\alpha(x) f(\alpha, x) d^4x \in \mathcal{H}_1. \quad (16)$$

The Grassmann algebra is then the direct orthogonal sum $\mathcal{G} = \bigoplus_{n=0}^\infty \mathcal{H}_n$ with $\mathcal{H}_0 = \mathbb{R}$ and \mathcal{H}_n being the closed linear span of all monomials $\hat{\xi}(f_1) \dots \hat{\xi}(f_n)$, $f_j \in \mathcal{H}_\mathbb{R}$. The Lagrangean corresponding to real white noise is

$$L_0 = \frac{1}{2} \sum_{\alpha, \beta} \int \hat{\xi}^\alpha(x) C_{\alpha\beta} \hat{\xi}^\beta(x) d^4x$$

and the normalized Berezin integral yields

$$\begin{aligned} \frac{1}{Z} \int e^{-L_0} \hat{\xi}(f_1) \dots \hat{\xi}(f_n) \Pi d\hat{\xi} &= 0 \quad \text{if } n \text{ odd,} \\ \frac{1}{Z} \int e^{-L_0} \hat{\xi}(f_1) \dots \hat{\xi}(f_{2n}) \Pi d\hat{\xi} &= \text{pf}(\omega_0(f_i, f_j)) \end{aligned} \quad (17)$$

where ω_0 is the symplectic form (14).

Both, the fermionic white noise $\xi^\alpha(x)$ and the Berezin field $\hat{\xi}^\alpha(x)$, are elements of a real Grassmann algebra which is supplemented by the symplectic form ω_0 . But there is an essential difference between these fields. Polynomials of fermionic white noise are numerical functions on a measure space, their expectation values are calculated by a numerical integration with a positive measure. Polynomials of the Berezin field $\hat{\xi}$ are elements of an abstract Grassmann algebra. To calculate expectations one needs a linear functional on this algebra – the "Berezin integral" – which cannot be reduced to a numerical measure.

For the real fermionic white noise ξ and for the real Berezin field $\hat{\xi}$ the algebra can be extended to a complex algebra in choosing complex test functions $f \in \mathcal{X} = \mathcal{L}^2(\mathbb{R}^4) \otimes \mathbb{C}^4$. Thereby the form ω_0 is extended to a \mathbb{C} -bilinear form $\mathcal{X} \times \mathcal{X} \rightarrow \mathbb{C}$. The use of complex test functions is convenient for the construction of Majorana and Dirac fields, and it is necessary for the Osterwalder–Schrader reconstruction. This simple extension to complex test functions should be clearly distinguished from complex white noise which is built up by two independent real white noise fields $\xi_{(1)}$ and $\xi_{(2)}$

$$\zeta(x) = \xi_{(1)}(x) + i\xi_{(2)}(x), \quad \bar{\zeta}(x) = \xi_{(1)}(x) - i\xi_{(2)}(x).$$

For random fields the bar $\zeta \rightarrow \bar{\zeta}$ means always complex conjugation. The resulting fermionic complex white noise is characterized by the expectations

$$\begin{aligned} \mathbb{E} \zeta^\alpha(x) \star \zeta^\beta(y) &= \mathbb{E} \overline{\zeta^\alpha(x)} \star \overline{\zeta^\beta(y)} = 0, \\ \mathbb{E} \overline{\zeta^\alpha(x)} \star \zeta^\beta(y) &= \mathbb{E} \zeta^\alpha(x) \star \overline{\zeta^\beta(y)} = C_{\alpha\beta} \delta^4(x-y). \end{aligned} \quad (18)$$

Usually fermionic complex white noise is constructed with the two point functions

$$\mathbb{E} \overline{\zeta^\alpha(x)} \star \zeta^\beta(y) = -\mathbb{E} \zeta^\alpha(x) \star \overline{\zeta^\beta(y)} = \delta_{\alpha\beta} \delta^4(x-y) \quad (19)$$

see [15] for the Berezin version and [8] for the random fields. Fermionic complex white noise with these expectations has no natural decomposition into two real fields.

IV. Euclidean Majorana and Dirac fields:

Since the structure of the Schwinger functions of Majorana fermions (10) is the same as the structure of the n -point functions of fermionic white noise (13), we can obtain Euclidean Majorana fields by a linear transformation of white noise. We start from real white noise ξ with the fermionic multiplication (or we take the Berezin field $\hat{\xi}$), but now we extend the test functions to the complex space $\mathcal{X} = \mathcal{L}^2(\mathbb{R}^4) \otimes \mathbb{C}^4$ as needed for Euclidean fields. Let A be a bounded linear operator on \mathcal{X} , then $\psi = A\xi$ is a random field with $\psi(f) = \xi(A^T f)$, where A^T is the transposed operator of A . The expectation values of ψ immediately follow from (13). They coincide with the Majorana Schwinger functions if $E\psi(f)\star\psi(g) \equiv \omega_0(A^T f, A^T g) = \omega_M(f, g)$. The operator A has therefore to satisfy the identity

$$A C A^T = SC \quad (20)$$

The solution which depends only on ∂ and therefore guarantees Euclidean covariance is

$$A = \alpha + \beta\partial \quad \text{with} \quad (21)$$

$$\alpha = \frac{1}{\sqrt{2}} \left[\frac{\sqrt{m^2 - \Delta} + m}{\sqrt{m^2 - \Delta}} \right]^{\frac{1}{2}}$$

$$\beta = \frac{1}{\sqrt{-2\Delta}} \left[\frac{\sqrt{m^2 - \Delta} - m}{\sqrt{m^2 - \Delta}} \right]^{\frac{1}{2}}$$

Here $\Delta = \sum_{\mu} \left[\frac{\partial}{\partial x^{\mu}} \right]^2$ is the Euclidean Laplace operator, and the pseudodifferential operators α and β are positive real operators (bounded if $m > 0$). Hence we can define a Euclidean Majorana field as the linear transform of real white noise

$$\psi_M = A \xi \quad (22)$$

The Berezin version of this construction is obviously $\tilde{\psi}_M = A \hat{\xi}$.

A Euclidean Dirac field ψ_D and its conjugate field $\tilde{\psi}_D$ can now be derived from two independent real white noise fields $\xi_{(1)}$ and $\xi_{(2)}$ or, equivalently, from two independent Euclidean Majorana fields by equations which are known from classical spinor fields on the Minkowski space

$$\begin{aligned} \sqrt{2} \psi_D &= \psi_{M(1)} + i\psi_{M(2)} = A\xi_{(1)} + iA\xi_{(2)}, \\ \sqrt{2} \tilde{\psi}_D &= C \psi_{M(1)} - iC\psi_{M(2)} = CA\xi_{(1)} - iCA\xi_{(2)}. \end{aligned} \quad (23)$$

The two point functions are easily calculated with the correct result

$$\begin{aligned} E \psi_D(f) \star \psi_D(g) &= E \tilde{\psi}_D(f) \star \tilde{\psi}_D(g) = 0, \\ E \psi_D(f) \star \tilde{\psi}_D(g) &= \omega_0(A^T f, A^T C^T g) = \int f^T(x) S(x, y) g(y) dx dy. \end{aligned}$$

The Dirac field (23) can also be given as a function of the complex white noise (18)

$$\sqrt{2} \psi_D = A \zeta \quad \text{and} \quad \sqrt{2} \tilde{\psi}_D = CA \bar{\zeta}.$$

This representation differs from the representations given in [8] where a white noise field with the two-point function (19) has been used. It has already been pointed out in ref. [8] that there is no unique way to define Euclidean Dirac fields, but the construction of Euclidean Majorana fields as given here seems to be unique.

The definition (22) implies that the Euclidean Majorana field is a real random field with respect to the involution

$$K\psi = A \overline{A^{-1}\psi} \quad (24)$$

For $d = 2, 3 \bmod 8$ dimensions there exists a real representation of the Euclidean Dirac matrices. Since the operator (21) A is then also

real, the involution (24) reduces to $K\psi = \bar{\psi}$, and (23) is the decomposition of the Dirac field into its real and its imaginary part (in the usual sense). For $d = 4 \bmod 8$ we have to refer to a reality condition with respect to the non-local operator (24). In all cases the representation (23) shows that the Euclidean Dirac field and its adjoint field are linear combinations of two real Majorana fields. All these statements have their obvious counterpart in the Berezin formulation.

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